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Making See: A Structural Analysis of Mathematical and in Particular Game-Theoretic Writing

ABSTRACT: "Narrative does not make us see," Barthes proclaimed in 1966. Narrative, in Barthes's analysis, does not refer to anything outside itself, but operates exclusively in the sphere of language, generating *sense*—signifiers, not things being signified. I use Barthes's position to shed some light on mathematical writing. I develop the hypothesis that mathematical writing, though it uses the form of narrative, is referring to something—namely, mathematical objects—and hence relies on truth conditions emanating from the things being referred to, which feeds back into how mathematicians use the narrative code. This investigation, on the one hand, extends the reach of narrative analysis by bringing it to bear as a window into mathematical practice; on the other hand, it brings out certain aspects of the tools of narrative analysis in new ways. One of the central findings is that for mathematical writing, the Barthesian terms often work out "under reversed signs." For example, in narrative fiction, as Barthes says, everything is functional by definition. In mathematical writing, instead, functionality has to hold as a necessary condition, which has the effect that in the end everything is functional again. I further argue that the specific referential nature of mathematical narrating leaves certain markers on the text—markers such as explicit reference to the act of "seeing," calls on the reader to get involved with the argument, and a multiplicity of grammatically differently marked voices—which I document in three articles that have become classics in game theory.

KEYWORDS: *Barthes's structural analysis, argument as narrative, mathematical writing, textual markers, multiple voices, free indirect style*

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The Main Hypothesis[1]

In his closing remarks of "Introduction to the Structural Analysis of Narrative" (1966), Roland Barthes says:

> Narrative does not make see, it does not imitate; the passion that may consume us upon reading a novel is not that of a "vision" (in fact, we "see" nothing), it is the passion of meaning, that is, a higher order of relation, which also carries its emotions, its hopes, its threats, its triumphs: "what goes on" in a narrative is, from the referential (real) point of view, strictly speaking: *nothing*; "what happens" is language alone, the adventure of language, whose advent never ceases to be celebrated. (27)[2]

Writing mathematics is distinctively not that: *it does refer to something.* There is a thing, a mathematical object, being referred to, and the mathematician's job when writing down her findings is to make that object come to life in the mind of her reader—to make the reader *see* that object.

I base this on simple Fregean terms. Frege in fact uses an example from mathematics to illustrate the difference that he establishes between the *sense* (German "Sinn") and the *referent* (German "Bedeutung") of a sign: If a, b, and c are the lines connecting the vertices of a triangle with the midpoints of the opposite side, "the point of intersection of a and b" is one sense and "the point of intersection of b and c" is another, but they have the same referent. These different senses, to Frege, indicate different ways in which the same point is referred to, and therefore, the proposition that one equals the other contains true insight. Mathematics, in its most abstract form, can be said to be the science that deals with the different ways in which a thing can be referred to, the different *senses* of a thing, and that studies how these different senses relate to each other. All mathematics springs from that. Within that endeavor, senses are given by definition.

Narrative, in Barthes's analysis, is concerned with the bringing into being of sense—signifiers, if one wishes, not things being signified. Storytelling, in Barthes's terms, does not refer to anything outside the story. What is brought into being is the story. Storytelling is, from this perspective, a performative act. Mathematical writing is not—at least not in the strict sense, that is, at least not unless some truth conditions that lie beyond what happens on the level of language are fulfilled.

The distinction I am aiming at is similar to François Recanati's distinction between *illocutionary acts in the weak sense* and *illocutionary acts in the strong sense* (214–16). In Recanati's analysis, the phrase "I declare the session open" is, in itself, only an illocutionary act in the weak sense, because to utter it is not sufficient to effectively open the session. It will open the session only if the person uttering the phrase is in the position to open the session (if he or she is the chairperson, the president, or whoever presides over the session), that is, if some truth conditions that are beyond the act of uttering the phrase are satisfied. If these truth conditions are satisfied, then uttering the phrase "I declare the session open" becomes an illocutionary act in the strong sense and will effectively open the session. Similarly, the difference between mathematics and narrative.

Here is a passage from John F. Nash's 1950 paper, "Equilibrium Points in *n*-Person Games":

> Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point. (49)

What makes the above passage a piece of mathematics is that it is true with respect to truth conditions that hinge on the properties of the things referred to (closed graphs and convex mappings). If these truth conditions were not fulfilled, the two phrases would be some sort of narrative, but not a piece of mathematics.

One point needs to be clarified. On the view that I defend, mathematicians are concerned with the bringing into being of sense in one dimension, namely, in coming up with and writing down new *definitions.* I take definitions, as well as narratives, as abstract artifacts, which, once they have come into being, belong to the real world. However, a book of *definitions* does not make a book of mathematics. The activity of mathematics starts when different *definitions* (different senses) are related to each other.[3]

Mathematical notation—and this can be extended to mathematical writing in general—has sometimes been compared to musical notation: it is inanimate as such, but comes to life in the mind of the reader. What happens is not the experience of seeing certain shapes on paper (the musical or respectively mathematical notation). It is the experience of coming to work out by oneself the things these shapes refer to—the music, or respectively the relations between mathematical objects.

Mathematical writing is *argument.* It is argument in its purest form, one might say—argument that relies on axiomatically established, conventionally agreed-upon truth conditions. However, it can, and in today's practice most often does, take the form of narrative; and when it does, we can use the structural analysis of narrative to investigate how writers of mathematics appropriate and use that form to communicate their results. This is the main hypothesis that I want to explore in this essay. In doing so, I rely on a notion of narrative as an *abstract sequence of events recounted by a narrator.*[4]

I base this investigation on the terms developed by Barthes. One main tendency runs through the results of my inquiry: often the Barthesian terms work out under "reversed signs," or in some otherwise transformed way. Most clearly, this comes out for the basic question of functionality: while in narrative fiction everything is functional *by premise,* in mathematical writing, functionality is a *necessary condition*—a condition that has to hold if the purpose of the operation, namely communicating some truth about mathematical objects, is served well. If something is not functional, it has to go, so that in the end—if all works out well—everything is functional again.

As I will argue in the more text-analysis-based part of this essay below, the specific character of mathematical text as referencing something (for example, mathematical objects and the relations between them) leaves certain markers on the text that identify it as mathematical text. I propose to isolate three such markers of mathematical text in general and a fourth that is typical for applied mathematical text. I finally demonstrate these four markers in three articles that have become classics in game theory: John F. Nash's "Equilibrium Points in *n*-Person Games" (1950), Michael Spence's "Job Market Signaling" (1973), and Robert Aumann's "Agreeing to Disagree" (1976).[5]

How Mathematicians and Game Theorists Use the Narrative Form

"Where shall we look for the structure of narrative?" Barthes asks. "Of course, in narratives," he says (2). "But how shall we oppose the novel to the short story, the folk tale to myth, drama to tragedy (it has been done many times) without reference to a common model?" (1).[6] Barthes concludes that one needs a deductive approach, and that the beginning is to be made by outlining the first terms of such a theory. I will, in the following four major sections, take up the four larger terms of investigation proposed by Barthes—the language of narrative, function, actants, and narration (voice)—and use them as a tool to investigate mathematical, and in particular game-theoretic, text.

The Language of Narrative: Using the Narrative Code

Narrative, like language, is for Barthes first of all a *system of sense* (4; "un système de sens"). For Barthes, narrative is an *extended phrase* (4; "une grande phrase"). "To understand a narrative," he says, "is not only to follow the unfolding of the story, but it is also to recognize in it different strata, to project the horizontal sequence of events that builds the thread of the story to an implicitly vertical axis; to read (or to listen) to a story is not only to move from one word to the next, but also from one level to the next . . . meaning does not sit at the end of the narrative, it runs through it" (5).[7]

Mathematical narration is different. What the mathematician is interested in—is aiming at—does sit at the end of the narrative: *a mathematical result.* There is no unique way of getting there, surely. There is no unique order of representation, but there is a strict logical order at the end of which sits a result. Any meaning that possibly arises outside that logical order—that is only hinted at or alluded to—is, by definition, outside of the domain of mathematics. For example, a certain implication that is suggested by an example or analogy, but is never verified by a proof, does not count as mathematics.

The writer of a mathematical text can, however, use the force of the narrative code, notably how different strata of the text interact, to enhance the means by which the reader comes to understand the results being communicated. She can use it on the level of the *representation* (the "Vorstellung," to use Frege's term). For example, the writer of a mathematical text might introduce an example at one point rather than another to help the reader see a particular property under this or that aspect, which might nourish his intuition and so help him understand a certain result more quickly or effectively. Still and all, this is just a vehicle, a sort of pedagogical device. A mathematical text might be well-written or not-so-well-written, which is, at least partly, a question of how well the writer, consciously or not, manipulates the narrative code. But the mathematics itself is untouched by that.[8]

I want to illustrate this with an example taken from Michael Harris's essay "Do Androids Prove Theorems in their Sleep?" (2012). Harris relates how in 1990, for the Festschrift of Alexandre Grothendieck's sixtieth birthday, Robert Thomason published a paper of almost 200 pages that was co-signed by his recently deceased colleague and friend, Thomas Trobaugh. In the introduction to that paper, Thomason expresses the need to explain why the paper should be co-signed by Trobaugh. Thomason recounts a dream in which "Trobaugh's simiculus" appeared to him and made a certain claim, which Thomason, upon waking up, immediately knew had to be incorrect. However, Thomason explains that working out the argument for why the claim was wrong "quickly led to the key result of this paper" (117). Harris, in his own paper, notes that this is a familiar plot (not for a mathematical paper but for storytelling): a ghost appears and leaves a message. "But what is actually going on?" Harris asks. "The word *key,*" he explains, "provides the key for our understanding of this paragraph, and the Thomason-Trobaugh article as a whole. . . . The word *key* functions here to structure the reading of the article, to draw the reader's attention initially to the element of the proof the author considers most important" (131, emphasis original). This is a fantastic observation. It makes us come to understand that Thomason, in addition to being an extraordinary mathematician, is also a gifted writer. It makes us come to realize how Thomason makes his readers read the paper in a certain way without explicitly instructing them; how he makes them think about the thing to be shown in a certain way by silently directing their thoughts and their vision. However—and this is the point that I am aiming at—the results reported in the Thomason-Trobaugh paper could stand without that narrative move. Thomason could have used a more conventional form of signposting, something in the style of: "The main result is given in section . . ." or "The key element of the proof is . . ." The use of the episode with the dream is a thoroughly narrative device. It fulfills the double functionality of explaining why Trobaugh's name should be on the paper and providing the reader with a structure that helps him to work through the paper. Maybe Thomason himself was not aware of the functionality of that passage when he opened the paper in that way. But he may well have been, in which case his readers, without noticing it, are in the grip of Thomason's narrative hand from the very beginning, when he claims to have the moral responsibility to explain why Trobaugh's name should be on the paper, and thus creates immediate suspense (a moral dilemma). There is, after all, nothing very surprising about Trobaugh's name appearing on the paper, given that Thomason and Trobaugh have been long-term collaborators and the paper draws on patterns of reasoning that they developed together. Whatever is the case, it is Harris's fine observation to have spotted the functionality of that passage in making the reader come to understand what is happening in the paper. But here again, it is Harris's fine observation as a critic that leads him to this analysis, not his abilities as a mathematician.

That the writer of a mathematical text can manipulate the narrative code in order to communicate her results is made possible by a certain predisposition in her readers: it works on the basis that we all—as human social beings—are natives in the language of narrative. We have been conversant in it since listening to our first stories as children. And therefore, when we come to read a mathematical paper, because it is presented to us in the form of narrative, we read it, consciously or not, through the lens of the narrative code. There is one fundamental difference, though. When we open a novel, we do not know what there is in it for us: it might lift us up, enlighten us, confuse us, or entertain us. Every mathematical text, instead, starts with one clear promise: *I have understood something, and I want to tell you what it is.* This is the very reason why the text exists, and from a pragmatic point of view, it pervades the reading of the text in every instance. The reader will interpret every move, at every level of the text, on that promise.

Function

Is Everything in a Narrative Functional?

"In the domain of discourse," Barthes says, "what is noted is, by definition, notable" (7).[9] This principle, of course, goes into the reading of the text, and the writer, in turn, takes it into account when she composes the text. In the domain of fiction, the ability of the writer to manipulate the expectations of the reader can be used to create suspense, surprise, or deception: "Even if a detail seems irrefutably insignificant and rebels against each function, it would nonetheless end up pointing out the sense of the absurd or useless," Barthes says (7).[10]

In mathematical writing, there is no room for the absurd or useless, for deception, or for leading the reader up a garden path. *Because mathematical writing is reference to something, the functionality of every element can be objectively determined.* It is possible to say definitively, for example, that this or that lemma should be taken out because it is useless for the rest of the argument.

There is then, on the level of functionality, a clear difference between fictional narrative and the narrative form as it is used for the communication of mathematical results: while in fictional narrative everything that is noted is notable *by definition,* in mathematical writing, because it is reference to something, functionality can be determined with respect to that something. However, the reader takes it for granted that the writer has respected the principle to report only what is relevant, and so he reads in good faith, taking up everything as being relevant. One can then say that the reader's predisposition to take in everything as being functional operates "under two different signs": while in the case of fiction functionality works by convention, in the case of mathematical writing it works on the basis of a contract between the writer and the reader, which obliges the writer to report only what is functional, and it is possible to determine whether the terms of that contract have been respected or not.

To say it game-theoretically: in fiction, functionality is one of the rules of the game; in mathematical writing, it is a property attained in equilibrium (but not guaranteed a priori).

Cardinal Functions and Catalyzers

Barthes, in accordance with the other French structuralists,[11] takes as the basis of his investigation the Russian formalist school, represented most forcefully by Vladimir Propp's analysis of the folk tale. The Russian formalist school, in Barthes's reception, regards as a functional unit (or *function* for short) "any segment of the story that presents itself as the term of a correlation" (7).[12] "From a linguistic point of view," Barthes says, "the function is obviously a unit of content: it is 'what an utterance means,' not the way it is made, which constitutes it as a functional unit" (7).[13]

Barthes—and in this he goes beyond the Russian formalist school—distinguishes two large classes of functional units:

1. *functions in the strict sense,* namely *correlates between acts,* and
2. *indices,* which, instead of referring to a complementary or consequential act, refer to "a more or less diffuse concept, which is nevertheless necessary to the story: personality traits concerning characters, information with regard to their identity, notations of 'atmosphere,' etc." (8–9).[14] Indices are signifiers, Barthes explains. Often, several indices point to the same thing being signified, and the order in which they appear is not necessarily important.

This basic categorization of functional units, Barthes suggests, can be used to classify narratives, with folk tales as very functional on one end and psychological novels as very indexical on the other. From this perspective, mathematical writing is certainly very functional.

Within the class of functions in the strict sense, Barthes distinguishes further between *cardinal functions* (or *nuclei*) and *catalyzers.*[15] Cardinal functions, in Barthes's conception, are "moments of risk"—moments when the narrative can turn this or that way. Catalyzers, on the other hand, saturate the space between cardinal functions: "The functionality involved is purely chronological (what is described is what separates two moments of a story), whereas the link between two cardinal functions possesses a double functionality, at once chronological and logical: catalyzers are no more than consecutive units, while cardinal functions are both consecutive and consequential" (10).[16]

Catalyzers, Barthes argues with reference to Jakobson, serve an essentially *phatic function;* they maintain the contact between the narrator and the reader. "Any notion which in the first place just seems like filling up the space, always has a discursive function: it precipitates, delays, or revives the discourse, it sums up, anticipates, and sometimes even befuddles: what is noted always appearing as being notable, the catalyzer constantly reproduces the semantic tension of discourse, does not cease to say:

there has been and there will be sense . . . Let us say that one cannot take out a nucleus without changing the story, but one also cannot take out a catalyzer without changing the discourse" (10).[17]

In mathematical writing, the succession of cardinal functions is the succession of definitions, lemmas, propositions, theorems. Within this, notably, on the level of proofs, there are micro-sequences of cardinal functions.[18]

Barthes's concept of catalyzers allows us to expose an important difference between narrative and mathematical writing: in mathematical writing, there is no room for catalyzers in their function of saturating the space between cardinal functions. Mathematics—and this is a classical position—deals with timeless truths, and therefore in its presentation does not seek to explore time.[19] Mathematical writing strives for brevity. No definition, lemma, proposition, or theorem is or should be included just to create the effect of time elapsing. The actants in a mathematical narrative are never called for unnecessary action—actions that do not advance the argument or do not qualify as "moments of risk." In mathematical writing, everything that pertains to the sphere of mathematics is, strictly speaking, a moment of risk.

However, while mathematics does not deal with time, the writer of a mathematical text, on the level of the representation, may well take into account that for the reader time is nevertheless elapsing—the reader might want to pause between two cardinal functions, or might find it useful to have a certain image or intuition before moving on to the next step in the argument. In a mathematical paper, there is, then, room for a discursive element that fulfills the phatic function that catalyzers serve in narrative fiction, namely to maintain the contact between the writer and the reader. This phatic function is fulfilled by all sorts of comments, remarks, and examples that the writer might insert in order to orient the reader; for example, a comment explaining why a proof is approached in this or that way, or linking an example to the general result. This form of discourse is often explicitly marked, for example by putting it in a special section labeled "remarks," or by a change of mode, tone, or grammatical voice (a switch from an impersonal style to a "we"; a direct call to the reader; etc.). The very function of such explicit marking is to tell the reader: "This is not part of the argument. It is a comment. You can relax here, I am just telling you what has been said, will be said, or another way of saying it." Discursive elements of this kind are not necessary to make up a mathematical paper. The degree to which a mathematical text will contain this kind of discourse is to a large extent a question of personal style, style convention in a particular field of mathematics, and style requirements of specific journals. But most importantly still: never should the writer of a mathematical text put in something that is not somehow useful, that literally just fills the space. The craft of mathematical writing is to (silently) take care of the phatic needs of the reader without giving up on the principle of maximal parsimony.

Indices Properly Speaking and Informants

Within the class of indices, Barthes distinguishes further between *indices in the proper sense*—implicit signifiers of some characteristic—and *informants,* which are direct signifiers: "Indices need to be deciphered: the reader has to learn to know a character or atmosphere; informants impart knowledge ready-made," he says (11).[20] For example, Barthes explains that when James Bond deposits a set of keys in the décolleté of a chamber maid, this constitutes an index in the proper sense (it signifies that James Bond is seductive and charming); when the age of a character is provided explicitly, this constitutes an informant.

These categories, Barthes says, are meant as analytical tools. One and the same segment can fulfill multiple functions. A character performing a certain action can function both as a catalyzer and as an informant.

In mathematical writing, indices appear frequently in the form of *examples.* Examples are indices, in a quite literal sense. They truly indicate only—a particular property, the way a solution works out, etc. They are the reflection of some more general fact without being, or proving, that general fact.

Informants—simple statements of facts—play out on different levels. They can become relevant for the argument or fulfill a purely phatic function only; and they might be more or less direct. For example: "There is a unique equilibrium" is a direct informant. "The derivative is strictly decreasing" is also an informant, but its significance for the sequel of the argument probably lies in something that is implied by that statement, such as that the function has a unique maximum and that therefore a unique equilibrium will exist. Mathematicians often use the second, more primitive, type of informant because the reader can easily work out the implication by himself, and because one informant can have many implications.

What is Said and What is Not Said: "Narrative" Implicatures

Not only what is said, but also what is *not* said, can be functional. Barthes does not explicitly state this principle, but he shows it at work in a passage from Ian Fleming's *Goldfinger:* the sentence "James Bond sees a man of some fifty years," Barthes explains, of course imparts the factual information that the man in question is of some fifty years, which provides a certain character trait that might be important later on. But another immediate signification of the utterance is that Bond does not know the man (7). Barthes identifies this fact, but he does not further analyze the mechanism at work here.

The narrative device used here, following Grice's terminology, is a *communicative implicature* (44) that thrives on the specific narrative mode and point of view of the novel. I want to call such an implicature, when enwrapped in a narrative, a *narrative implicature.* In the example at hand, the implicature thrives on the narrative's point of view that the narrator says what the main character knows. It does not logically follow from the fact that the man is of "some fifty years" that Bond does not know the man, but this is implied if we interpret the line under the additional assumption that the author would have told us the man's name had Bond known the man.

Our ability to understand the meaning of communicative acts in our everyday lives crucially relies on our ability to do this sort of inference. We do it effortlessly, without having a clear theoretical model of the mechanism behind it in mind. The narrative code draws on this aspect of our general linguistic competence.

Mathematical writing can draw on this competence as well. In narrative fiction, the effect of such an implicature is the greater and more powerful if it is left invisible (that is, when it is transported by the specific point of view of the narrative rather than spelled out explicitly). But in mathematical writing, the writer sometimes quite insistently spells out what is already latent due to an implicature that thrives on the semantic-pragmatic form in which a statement is delivered; the writer actually has to do so in order to make the information part of what is mathematically said. And, importantly, such explicit spelling-out of what is already latent *will not spoil the narrative,* as would, for example, the note "Bond did not know the man." Instead, it will quicken the narrative, because it will help the reader to reach the result and its implications more rapidly, and may provide new information that can in fact not be provided by the implicature. Here again, then, is a difference between fictional narrative and mathematical writing, not so much in principle as in the sign under which a specific function is made use of.

Here is an example from Robert Aumann's 1976 article, "Agreeing to Disagree":

> **PROPOSITION.** Let $\omega \in \Omega$ and let q_1 and q_2 be numbers. If it is common knowledge at ω that $\boldsymbol{q_1} = q_1$ and $\boldsymbol{q_2} = q_2$, then $q_1 = q_2$ (1237)

This statement harbors an implicature, namely that anything *less than* common knowledge of q_1 and q_2 at ω is *not sufficient* for the conclusion to hold true, or can so far not be shown to be sufficient.

This can be analyzed as an implicature that thrives on the communicative maxim of *quantity,* "Make your contribution as informative as is required (for the current purposes of the exchange)" (Grice 45), which in mathematical communication acquires the more specific form that sufficient conditions are to be stated in terms of the least restrictive sufficient conditions that one is able to identify. Certainly, what is suggested by such an implicature does not follow logically from the statement of the result, but is implied only if one supposes that a convention is in place that the author would have given less restrictive conditions if he had identified less restrictive conditions. But such a convention is outside the domain of mathematics; it pertains to the domain of language. So mathematically, the content of the communicative implicature is not valid. Mathematically, it is not there.

Aumann takes care of his readers' wish for mathematically precise information. He explains that the result fails when people have simple first-order knowledge (instead of *common knowledge*) of q_1 and q_2, and he demonstrates that by an example ("Agreeing to Disagree" 1237).

An Excursion: *Showing* and *Telling* in Mathematical Writing?

What is said versus what is left unsaid in a narrative is closely related to the more general question of *showing versus telling.*

Can a showing–telling opposition be made meaningful for the investigation of mathematical writing? Is the main argument of a mathematical paper—the succession of definitions, lemmas, and theorems—a form of showing or telling? I would like to defend the hypothesis that it is a form of *showing*—a form of showing that necessitates the active participation of the reader in working out what is happening on the level of the main actants, the mathematical objects.

For example, in the passage from Nash's article quoted above ("Since the graph is closed . . ."), from a grammatical perspective it is "we," the readers and the writer, who are doing something: "we infer." But what we infer follows from the property of the "graph being closed" (a cardinal function) and the "image of each point under the mapping to be convex" (a cardinal function). The writer makes us, the readers, watch this acting-out of the cardinal functions to give another, new, cardinal function: "the mapping has a fixed point." The indication that what is going on is by "Kakutani's theorem" operates in a different sphere of action than that of the cardinal functions: it happens on the level of the interaction between the writer and the reader and their being part of the academic community. It is like a voice-over telling us, reminding us, that what is going on in the particular case is the reflection of a more general relation that was first observed by Kakutani. Finally, and this now is Nash's contribution, it follows from the existence of a fixed point that the game has an equilibrium point.

What makes mathematical papers hard to read for outsiders, I would claim, is precisely the fact that what is going on is *displayed* only—set out to be watched—and is not being told.[21] To understand what is put out there to be watched—to be able to *see*—one needs some ready-made background knowledge, and therefore, without a certain background knowledge, one cannot watch.

Of course, what shines through this debate is the more fundamental question: *What is showing?* What is meant by that metaphor? *Showing* is often understood to refer to an advancement of a narrative by having the actants in the narrative do something. The defining feature of *showing,* it seems to me, is that the advancement of the narrative does not come from the observed action itself but from what this action reveals (a certain motive, character trait, etc.).

I want to explain this with reference to the concept of showing that emerges from Wayne Booth's discussion of that term. Booth, arguing against the "clichés about the superiority of dramatic showing over mere telling" ("Distance" 61), quotes the following blurb from a Modern Library dusk jacket of Salinger's *Nine Stories:*

> The author does not tell you directly but you find out for yourself from their [the characters'] every word, gesture, and act. (quoted in Booth, "Distance" 61)

"That this is praise," Booth says, "that Salinger would be in error if he were found telling us anything directly, is taken for granted" (61). What is essential in this passage—and I would argue constitutive for the difference between showing and telling—is the opposition between "the author does not tell you directly" (telling) and "you find[ing] out by yourself" (showing).

Showing, I would like to propose, means to impart particular information about cardinal functions or actants in the narrative by *exploiting knowledge that the reader brings to the narrative from outside*. To my mind this is the very reason why *showing* in narrative fiction is "easier" on the reader: because he does not need to learn something in the world of the narrative but can instead draw on a "vision" that is readily available to him. In narrative fiction, the material readily available to the reader is his real-life experience. *Showing* in narrative fiction is such an effective and powerful device because that experience of the reader is so rich, complex, and diverse—and, most importantly, it comes for free. In mathematical writing, the material readily available is the knowledge that the intended reader is supposed to have as a member of the mathematical community or a certain subfield of it. But that knowledge does not come for free; it has to be acquired laboriously. As a consequence, depending on how saturated and active that knowledge is, the reading may be more or less easy. For someone who already knows what a fixed point is (and knows Kakutani's theorem), following the proof that Nash gives us—*shows* us—for the existence of an equilibrium in n-person games is indeed easy.

Extending the concept of *showing* proposed above, I would propose that *telling* means imparting narrative-specific information to the reader. Commentary can thus be classified as a form of *telling.* In commentary, the mathematical objects are not acting out some relation between them; instead, the writer is directly addressing the reader, giving him some complementary information, spelling out some implicit assumption, telling him what has been going on or will go on, what he was supposed to see, why it has been shown to him, why it has been shown to him in a particular way, why the problem is interesting, etc. Precisely for this reason—because it is a form of telling—commentary in mathematical writing is easier to understand, and therefore may offer a moment of pause for the reader.

Again then, there is a certain inversion under which the same basic principle manifests itself in one way in narrative fiction and in the opposite way in mathematical narration: while in narrative fiction, showing has a tendency to be easier on the reader than telling, in mathematical narration, telling is often easier on the reader than showing.

Aumann's paper again provides a good example. In the statement of the proposition quoted above, it is not explicitly spelled out that common knowledge of q_1 and q_2 at ω has to be established *in virtue of the individuals' common knowledge of their information partitions.* A reader who works through the paper (and notably the proof of the result) understands that. Aumann knows, however, that a reader might lose sight of this, and therefore he explicitly alerts his readers that "worthy of note is the implicit assumption that the information partitions are themselves common knowledge," which, he adds, constitutes no loss of generality because "included in the full description of a state ω of the world is the manner in which information is imparted to the two persons" (1237).[22] Aumann's comment here is truly phatic. For the reader, this kind of *telling*—explaining and pointing out—is not a burden but actually makes the reading easier, because what was put out there to be watched operated on a certain hidden "character trait" of the state of the world.

What Booth says about narrative fiction is decidedly also true for a piece of mathematical writing: the writer's "artistry lies not in adherence to any one supreme manner of narration but rather in his ability to order various forms of telling in the service of various forms of showing" (*Rhetoric of Fiction* 16).

The Unraveling of the Narrative and the Real Time

In narrative fiction, the order in which events are related/revealed to the reader is not necessarily the order in which things happened in the world of the fiction. But irrespective of how the story is related—there might be flashbacks, some elements might be related through the narration of a character, etc.—there is a story-immanent real time, and at some point (whether before, during, or after writing) the writer has to set this straight in her mind to achieve consistency. Of course, in more experimental approaches, the writer might choose a different model of time according to which the story-immanent time unfolds—for instance, it might unfold as a loop. (Even if this is so, the author is then playing with the reader's expectations, which will lead him to initially read the narrative as if there is a story-immanent real, naturalistic timeline.) In any case, she has to make sure that the unfolding of the story remains within the constraints set by that model. The reader needs to be able to reconstruct the order in which things happened in the world of the fiction. In fact, it is one of the reader's predispositions to always try to reconstruct that order.

In mathematical writing, this relation appears under reversed signs. In real time, in everyday life, a sequence of events unfolds through which the researcher comes to understand the things that she will eventually consider important enough to communicate. (For example: She is sitting in a talk . . . She realizes that . . . She tries to prove that . . . By using . . . But that fails, because . . . She tries some other approach . . . She thinks that she has another proof, but that fails again . . . She understands why it failed . . . And this finally leads to the converse of the result that she wanted to show in the first place.) But this is not necessarily or even regularly the sequence of events that will be shown to the reader. Indeed, today's convention of how to write mathematics requires that it not be. In mathematics, one usually tries to find the shortest logical deductive representation. Of course, one could write a mathematical paper along the real line of how one has come to understand the result, and there is necessarily some logic to that sequence of events; for otherwise this sequence could not have led to the result. But usually, this involves too much redundancy, too many failed attempts and wrong intuitions, etc.[23] All of these inefficiencies are removed from the form in which the cardinal functions are ultimately related to the reader. The order in which the sequence of events happened in real time is completely abstracted from. And most importantly: the reader does not care about it; the reader is not driven by an ambition to reconstruct that order (as is the reader of narrative fiction).

Actants

Classical structural analysis attempts to trace character back to *actants* (or *agents*), that is, to define characters by their participation in the sphere of actions. In other words, characters are classified according to some typology based on the actions assigned to them by the narrative, not based on psychology.

Functions and Spheres of Action

With his discussion of catalyzers, Barthes implicitly recognizes functionality in two different *spheres of action:*

1. what is happening in the narrative (what the actants in the narrative do), and
2. the transaction between the narrator and the reader. (9–11)

According to Barthes, in narrative fiction, an entirely phatic function can be expressed by having the actants in the story do something—which is precisely what he calls a *catalyzer.* One might say then, with Barthes, that in narrative fiction, the sphere of action of the actants in the narrative can *invade* the sphere of action between the narrator and the reader.

Important to note is that *a priori* there is nothing in the outside form of an action taken by the actants in the narrative that would identify it as a cardinal function or a catalyzer. "He lit a cigarette" can be a cardinal function or a catalyzer. Whether it is one or the other will reveal itself only in the sequel of the narrative. The experience of reading fiction, to some extent, depends on the fact that the reader cannot always immediately tell whether some action is a cardinal function or not, and the writer might purposefully play with that. Also important to note is that it is precisely this kind of unnecessary action in the narrative (the emptiness of which might become evident only later on) for which there is no room in mathematical writing. In mathematical writing, the reader wants to know immediately what is a cardinal function and what is not, and the writer, if she acts in the service of her reader, will do her best to mark that clearly.

Interestingly, the inverse invasion happens in mathematical writing: in mathematical writing, it is regularly the case that the narrator and the reader interfere with what is happening in the narrative (the development of the mathematical argument). Of course, such invasions can also happen in narrative fiction. But when they happen there, they often serve the phatic function of negotiating the contact between the narrator and the reader, or they may help the reader to orient himself in the story. This might take the form of something like "In the last chapter, we saw that our hero . . ." What's so particular about invasions of the narrator and the reader into the sphere of action in mathematical narration is that it *negotiates cardinal functions*—that it actually brings forward a movement on the level of cardinal functions. This is most audible in proofs. One clearly hears it in the passage from Nash's article, quoted above

("Since the graph is closed . . ."). In that passage, not only does the writer urge the reader to infer something, but this movement of thought advances the narrative: it establishes that a fixed point, and hence an equilibrium, exists.

Similarly, and more engaging still, is the following passage from Spence's article:

> We begin by noting that, under the assumptions, the conditional probability that a person drawn at random from the population has a productivity of two, given that he is a man (or she is a woman), is the same as the unconditional probability that his productivity is two. Sex and productivity are uncorrelated in the population. Therefore, *by itself,* sex could never tell the employer anything about productivity.
>
> We are forced to the conclusion that if sex is to have any informational impact, it must be through its interaction with the educational signaling mechanism. But here again we run up against an initially puzzling symmetry. Under the assumptions, men and women of equal productivity have the same signaling (education) costs. . . . Hence, again we appear to be driven to the conclusion that sex can have no informational impact. But the conclusion is wrong, for an interesting reason.
>
> The opportunity sets of men and women of comparable productivity are *not* necessarily the same. To see this, let us step back to the simple educational signaling model. (369–70, emphasis original)

The narrator is most decidedly present, telling the reader what to do, what has to be concluded, and what can be seen. In both of these passages, the narrator is not just there to support the reader in seeing what is going on, but the narrator and the reader *act together* to work out a result that is somewhat dormant in the mathematical objects. The intervention of the reader and the writer is not just phatic but brings forward some movement on the level of the cardinal functions.

Narrative within Narrative

Applied mathematical narration has a third level of action, beyond the sphere of the narrative and the transaction between the narrator and the reader: it contains narrative within narrative, where the latter is the story of the problem to be studied. In a game-theoretic paper, that problem is the *game.* No matter how theoretical a game-theoretic paper is, even if its aim is to study the purely formal properties of some solution concept, there are always, at least silently, the people in the model ("*n* players," in Nash's paper; "individual 1 and 2," in Aumann's paper; "the employer" and the "job applicants"—"men and women," in Spence's paper). And, importantly, they too interfere with what is going on in the main argument.

An article by Elon Kohlberg and Jean-François Mertens contains a poignant example. In the relevant passage, Kohlberg and Mertens try to convince their readers that a certain equilibrium, namely (*T,R*), is not a good prediction for the solution of the game under study:

> Observe, however, that (*T,R*) is strategically unstable: player II knows that [player] I will never choose *B,* which is dominated by *T* . . . so if II sees he has to play, he should deduce that I, who was supposed to play *T* and was sure to get 2 in this way, certainly did not choose *B,* where he was sure to get less than 2; player II should thus infer that I had in fact played *M,* betting on a chance to get more than 2 (and on the fact that II would understand this signal); and so player II should play *L,* and hence player I should play *M,* deviating from the equilibrium prescription. . . .
>
> We see then that conformity with backwards induction, while being necessary for strategic stability, is not sufficient. (1007)

In this passage, all three levels or spheres of action—the actants on the level of cardinal functions (the game-theoretic argument), the reader and the writer, and the players in the game—interact. The three levels of action are connected by some form of "looking on": in the game, "if II sees he has to play," he should deduce that "I had in fact played *M*" and has to draw certain conclusion from that. We, as readers, see this seeing in the world of the game, and we observe this observing and reasoning in the world of the game. And from that, "we see" some movement on the level of the cardinal functions: that *backwards induction is a necessary but not sufficient requirement for strategic stability.*

Narration (Voice)

Narration in fiction usually operates under the convention that there is unity in the grammatical voice that is giving the narrative (at least within certain well-defined units, parts, chapters, or paragraphs). Examples that show a disrespect for that principle are rare and have given rise to debate.[24] The narrative can be given by an impersonal narrator or a narrator who is anchored in the narrative, a narrator who does or does not know more than the characters or a specific character, a narrator who is thought of as being identical with the author or not, a narrator who is reliable or not, etc. In any case, with the exception of more experimental approaches that have the narrator change from chapter to chapter or paragraph to paragraph, there is usually unity of the grammatical voice.

Multiple Voices and Polyphony

Writers of mathematical papers—and this is foremost an empirical observation—do not respect unity of the grammatical voice. It is not rare to find papers that start out in one voice (for example, an impersonal style as marked by the use of "one" or passive forms) but then flip into another voice (for example, a "we"), or flip back and forth between different voices. I will document this in detail in the three game-theory papers in the final major section of this essay.

This splitting up of voices, it seems to me, often goes unnoticed—unnoticed not only by the reader, but also by the writer. The reader does not notice the flip from one voice into another as an inconsistency (and he therefore doesn't notice it at all) because he automatically attributes some *functionality* to that flip; he interprets it as the author's functionalization of different registers of the text, different levels of personal involvedness or timelessness, or different strata of the text. He immediately recognizes: "Aha, this is Aumann talking in the authorial voice that exposes the result"; or "Now, this is Aumann talking as a member of the academic community"; or "And this is Aumann talking as Aumann, sharing his personal interpretation of the result." No one blames the writer of a mathematical text for flipping from one voice into another if, in the flow of the text, this sort of functionality can be recognized (if it is noticed at all). Barthes's first principle resonates here: the reader takes in everything as being functional. The writer, on her side, does not notice that she is moving from one voice to another because her attention is so absorbed by the task in which she finds herself involved—getting the results across—that any means that present themselves to her as stylistic devices (any twist or phrase that she has heard or read somewhere) come in handy, so that she is willing to take them up when convenient.

At the same time, different grammatically marked voices or modes tend to be polyphonic. The same instance of "we" can sound like an authorial "we," "you and I," "the reader and the writer," "we, the academic community," or "I, the writer, in agreement with the editors and the reviewers of this paper."

The tolerance of mathematical writing toward multiple narrative voices that are "locally" pragmatically interpreted, in my account, thrives on the basic condition of mathematical writing in that what is said has to be supported by truth conditions fulfilled by the things being referred to. In narrative fiction, the unity of the fiction needs to be maintained, and unity of voice seems to be one of the means by which this is achieved.[25] In mathematical writing, instead, no fiction needs to be maintained. Everything is true, or has to stand the test of being true. Everything is interpreted under the one basic assumption that the writer has gained insight into some truth and wants to communicate that truth to the reader.

Free Indirect Style

In game-theoretic papers, the presence of the actants in the game world opens the possibility for the use of *free indirect speech* or *free indirect style* (see, for example, Moretti, and Wood). "Free indirect style," as James Wood says, "is at its most powerful when hardly visible or audible: 'Jen watched the orchestra through stupid tears.' . . . What is so useful about free indirect style is that in our example a word like 'stupid' somehow belongs both to the author and the character; we are not entirely sure who 'owns' the word. . . . Thanks to free indirect style, we see things through the character's eyes and language but also through the author's eyes and language" (10–11).

Kohlberg and Mertens's article again provides a perfect example of this in mathematical writing:

> We saw that the "bad" equilibrium "2,2" was sequential; however, it is no longer sequential in the above presentation of the same game. (1008)

"Bad" is not a formal category for an equilibrium. Through the word "bad," the players in the game talk. We, as readers, immediately understand which equilibrium is being referred to because earlier in the paper (see the passage quoted above) the writers have made us think about the game through the eyes of the players in the game. The equilibrium 2,2 (which is the same as the one referred to as (T,R) in the passage cited above) is "bad," that is, unfortunate or inconvenient, *from the point of view of the players in the game,* who both strictly prefer some another equilibrium. But the equilibrium is actually also "bad" from the programmatic point of view that the writers defend (it is an equilibrium that does not pass the test of backward induction in every possible tree representation of the game), and the main message of the paper is in fact that "a good concept of 'strategically stable equilibrium'" should throw out equilibria of the kind of (T,R) (Kohlberg and Mertens 1004). The opposition "good concept–bad equilibrium" makes it immediately clear for the reader how the example connects to the main theoretical point of the paper, namely that a "*good* concept" has to rule out a "*bad* equilibrium." This is, to use Wood's terms, free indirect style at its most powerful: "we see things through the character's eyes and language but also through the author's eyes and language."

A General Pattern

One more general lesson emerges, then, from this view into the writing of mathematics through the prism of Barthes's terms of inquiry: often a condition that holds for narrative fiction is found to hold for mathematical writing "under reversed signs," or in some otherwise transformed way:

- In narrative fiction, everything that is noted is functional by premise; in mathematical writing, functionality has to hold as a necessary condition.
- In narrative fiction, the actants in the narrative might engage in unnecessary action between two cardinal functions in order to negotiate the contact between the narrator and the reader (catalyzers); in mathematical writing, there is no room for such superfluous acting.
- In narrative fiction, spelling out an implicature spoils its effect; in mathematical writing, what is suggested by an implicature needs to be spelled out, and demonstrated, in order to make it enter into the realm of what is mathematically on the record.
- In narrative fiction, *showing* is often easier on the reader than *telling;* in mathematical writing, *telling* is often easier on the reader than *showing.*
- In narrative fiction, the reader cares about the sequencing of cardinal functions (and constantly tries to reconstruct it); the reader of a mathematical narrative does not.

- In narrative fiction, what happens in the narrative invades the sphere of action between the narrator and the reader; in mathematical writing, the narrator and reader frequently invade the sphere of action of the narrative, not just on a phatic level, but actually to bring forward a movement on the level of cardinal functions.
- In narrative fiction, unity of voice seems to be needed to enhance the effect of the unity of the fiction; in mathematical writing, because truth has to be established with respect to something beyond the text, the narration can be transported by different voices that are locally pragmatically interpreted.

This reversal of relations, so I would defend, reflects the more fundamental difference between these two modes; namely, that mathematical narration is reference to mathematical objects that rely on truth conditions that are beyond the text, beyond the narrative, beyond the act of putting words together, and beyond the domain of language. In the following, and final, part of this essay, I want to demonstrate that this referential nature of mathematical writing leaves some (linguistic) markers on the text.

Textual Markers in Three Classical Papers in Game Theory

It is hard to find textual markers that definitively identify realist fictional narrative *as* fictional narrative, since many of their textual features are also present in other forms of narrative, like the narration of true historical events. What ultimately identifies a work of realist fiction are institutional markers that lie outside the text, such as the designation of a text as a "novel" on the cover of a book.

Based on a close reading of mathematical text, I postulate that mathematical writing carries the following three textual markers:

- references to the act of *seeing* ("now we see," "one can see," "it can be seen," "as we have seen");
- instructions to the reader, often clearly identifiable on the linguistic level, because marked as imperatives: callings to imagine ("let," "take," "suppose"), instructions to manipulate objects ("add," "substitute") or invitations to make movements of thought ("observe," "notice," "realize"); and
- a multiplicity of grammatically differently marked voices or narrative modes ("I" and "we"; "one" and "we"; or different strata of "we").

These markers, it should be noted, are concerned purely with the narrative code and language. I abstract from the obvious fact that mathematical text may, and in most cases nowadays will, contain symbols and formulas.

These imprints on the text, to my mind, reflect the very nature of mathematical text as referring to something, making it the very purpose of the text to bring this something (mathematical objects and relations that hold true between them) come to life in the mind of the reader. They seem to echo a dialogue between the writer and the reader: every invitation to “see” something, or the mere observation that it “can be seen,” assumes someone who is doing the seeing, and also someone who has seen it before. Every imperative silently assigns someone the role of the one who gives the order, and someone the role of the one who receives it. In the flow of a text in which “I” appears as the main narrative voice, “we” automatically becomes “you and I” or the entire community who reasons together. It is not rare to find two or all three of these markers embodied in the same linguistic token, in the same word or phrase—for example, in an outburst into “Now we can see . . .” at the climax of a text in which, up to that point, the use of any pronoun has been avoided.

In addition to these three markers, applied mathematical text, and game-theoretic text in particular, has one more distinct trait: it contains narrative within narrative. In addition to the mathematical argument, there is always the “story” of the model; in a game-theoretic text, this is the “game” being studied. There are always the actors of that story, whether they appear in a quite concrete scenario (the “job applicants” and the “employer”) or in a very abstract way (“*n* individuals,” “individual 1 and 2”). This narrative-within-the-narrative opens further possibilities, like interventions of the writer and the reader into what happens in that narrative-within-the-narative, or free indirect style through which the actors in that narrative-within-the-narrative talk.

In the following, I demonstrate these four textual traits in three papers that have become classics in the theory of games.

John F. Nash, “Equilibrium Points in *n*-Person Games” (1950)

Nash’s paper barely makes one page. In that one page, it contains all the features that I claim to be typical for mathematical writing. It offers an example of an impersonal style as the main narrative mode, and the use of “we” when it comes to formal arguments.

The paper opens in a passive, impersonal style: “One may define a concept of” (48); “any . . . may be regarded as” (49). “We” appears when the proof starts: “From the definition of . . . we see that”; “by using . . . we see that”; “since . . . we infer from . . . that” (49). The “we,” in contrast to the impersonal style, becomes polyphonic: “We” can be read as the author’s “we,” “you and I,” the writer and the reader together, or the entire scientific community to which they belong. In a footnote the author appears as a real person: “The author is indebted to . . . for suggesting the use of” (49). The third-person “the author” marks a distinction between the flesh-and-blood individual John F. Nash, and the narrator, who acts as a member of the scientific community.

The paper consists of a single result. The proof of this result is *not* separated from the main text (as it could be, for instance, in a special section labeled “proof”). Instead, it is, I would argue, linguistically marked: the move from the “one” to the “we” takes on the function of indicating a change in register, announcing the start of the proof. In the beginning, it is just “one” (as if saying “anyone” may define this or that). The appearance of “we” is like a call on the reader letting her know: “We have to do some work here together.” In the end, the move from the “one” to the “we” will have come to stand for a change in the state of knowledge of the reader: before, "one" has defined something; now, "we" have seen and understood something together.

Nash instructs politely and indirectly: "One may define" (48); "any . . . may be regarded as" (49).

Both the impersonal style and the "we" operate through some form of "seeing": "any . . . may be regarded as" (49); "from the definition of . . . we see that" (49); "by using . . . we see that" (49).

The story of the characters in the game appears in abstract, technical terms, but in fact opens the narrative: "One may define a concept of an *n*-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the *n* players corresponds to each *n*-tuple of pure strategies, one strategy being taken for each player" (48). Indeed, the reader is called upon to envision that story and to translate it into a mathematical representation.

Michael Spence, "Job Market Signaling" (1973)

Spence's article offers an example of an "I" as the main narrative voice and a "we" when it comes to formal arguments. This change of voices structures the paper into three parts, each marked by one dominant voice, giving a pattern of: "I"–"we"–"I." The article opens with the author's "I" explicitly appearing as the giver of the communication:

> The term "market signaling" is not exactly a part of the well-defined technical vocabulary of the economist. As part of the preamble, therefore, I feel I owe the reader a word of explanation about the title. . . . One might accurately characterize my problem as a signaling one, and that of the reader, who is faced with an investment decision under uncertainty, as that of interpreting signals.
>
> How the reader interprets my report of the content of this essay will depend upon his expectations concerning my stay in the market. If one believes I will be in the essay market repeatedly, then both the reader and I will contemplate the possibility that I might invest in my future ability to communicate by accurately reporting the content of this essay now. (355)

In fact, the author and "the reader" are recruited into the roles of the main characters of the theoretical problem to be studied; they are projected into the narrative-within-the-narrative.

"We," like in Nash's article, appears when the more formal argument starts: "In what follows, we shall make the assumption that" (358). Similarly, as in Nash's article, this change of voice signals a change in register. It is as if saying: "Beware, here starts the serious part. So far you were free just to listen. From now on, if you want to follow, you will have to do some work too." This "we" is polyphonic. It can be an author's "we," which in contrast to the author's "I" signals that the author is writing in a different register. But it can also be a "we" which, in contrast to the "I," now includes the reader: "we can imagine"; "we can think of" (360); "we shall assume that" (361); etc.

Proofs, as in Nash's paper, are not separated in special sections from the rest of the text but are rather introduced by that change in register: the appearance of this "we."

There is one instance of "we" that is *not* compatible with an inclusive "we" but that has to be read as the author's "we": "We summarize the underlying data of our numerical example in Table I" (361). A reader, I would argue, can be envisioned as constructing or reconstructing an argument. In that sense, the reader can be included in abstract actions like *supposing, assuming,* or *seeing,* and maybe even *summarizing.* But the reader cannot be included in "we summarize in Table I." Summarizing *in Table I,* that is, creating the physical shape of Table I, is something that happens before the reader has anything to do with the text. To push this to an extreme, if the phrase were to read: "We can summarize *as in* Table I," it could, I would argue, include the reader, but "we summarize in Table I" cannot.

Though "we" is the dominant voice in the middle part, the author's "I" is not altogether absent: "I find it most useful to think in terms of" (360); "I propose to discuss" (361); "I shall use"; "I shall refer to . . . as" (369). But clearly, "I" is reserved for comments on approach that are of a more conceptual nature or concern a choice of presentation, while "we" is used for negotiating the strictly formal argument. Similarly, the reader is addressed as "the reader" when additional, optional comments or interpretations are offered: "the reader will realize that" (363); "the reader may wish to think in terms of" (369); etc.

Spence instructs politely using a plural imperative "we": "Let us suppose that" (361); "let us assume that" (369); etc., which can be read as calling on the reader for some joint action but is also compatible with an author's "we" who politely asks to be allowed to "suppose" and "assume." Sometimes, Spence addresses the reader by singular imperatives: "Suppose that"; "consider" (362); "notice that" (364); "recall that" (365); etc.

The faculty to "see" is explicitly called upon: "To see this, let us step back to the simple educational signaling model" (370); "As we saw in the educational signaling model" (372); "Looking at this situation from outside, one might conclude that" (373); "We have looked at" (374).

In the main part of the article, in which Spence takes signaling in the job market as the paradigm case, the "employer" and the potential "employees" become the main actants. When it comes to demonstrating results, it is often the people in the model who do something, while the reader watches them: "Given the offered wage schedule, members of each group will select optimal levels for education. Consider the person who will set $y<y^*$. If he does this, we know he will set $y=0$ because education is costly, and until he reaches y^*, there are no benefits to increasing y, given the employer's hypothesized beliefs" (362).

Robert Aumann, "Agreeing to Disagree" (1976)

Aumann's article offers an example of an authorial "we" as the main narrative voice and the use of imperatives when it comes to demonstrations.

The first appearance of "we" is quite particular: "We publish this observation with some diffidence" (1236). This "we" cannot be understood in the sense of "you and I," the reader and the author. The reader, definitely, does not publish. The reader might reason with the author or follow the author, but he stays on the receiving end of the transaction. The "we" here is polyphonic but stays on the giving side of the transaction. It is the author or the author, the editors, and the reviewers together. "We publish with some diffidence" has the potential to work as a dynamic element: it can be understood as evoking the whole process of publication, as if saying "We, that is, I in discussion with colleagues, the editors, and the referees, we have thought about it, and after some debate, we came to the conclusion that it is worth publishing the result."[26] Later in the text, "we" keeps an authoritative tone, but is closer to a standard author's "we" and does not necessarily carry in itself this editorial voice (though it remains compatible with it): "When we say . . . we mean" (1236); "we will show below that" (1237); "Of course we do not mean" (1238); "our result implies that" (1238).

Some instances of "we" can be read in the sense of "you and I," the reader and the writer together: "The result is not true if we merely assume that" (1236); "we have" (1237); "we get" (1237). On the last of the three pages that make up the article, the author appears as a real person: "In private conversation, Tversky has suggested that" (1238). *Who was in "private conversation" with Tversky?* Aumann, of course—Aumann the real person. Finally, the author appears in the first person singular: "It seems to me that the Harsanyi doctrine is implicit in much of this literature" (1238).

The statement of the main theorem and its proof are put in separate sections labeled as "proposition" and "proof," respectively. A preparatory result, instead, is merely set apart in a separate paragraph, its proof introduced by the characteristic "To *see* that" (1237, my emphasis).

When it comes to the formal argument and demonstrations, Aumann explicitly instructs: "Call the two people 1 and 2"; "formally, let" (1236); "write"; "to see that . . . let . . . and call"; "suppose now that"; "consider next" (1237); etc. The use of imperatives naturally goes with—and follows from—the authorial "we."

The article, like Nash's and Spence's, starts with the narrative-within-the narrative: "If two people have the same prior, and their posteriors for a given event *A* are common knowledge, then their posteriors must be equal"—which in fact summarizes the main result. These people are even given names, even if very abstract names. Actually, the reader is instructed to give them those names: "Call the two people 1 and 2" (1236).

Perspectives

I have chosen these three papers not only because they are seminal contributions in game theory but also because they seem to me to exemplify a typology of mathematical papers (as far as single-authored papers are concerned) with three large classes:

1. Passive, impersonal style as the main voice, and "we" for formal arguments and proofs (Nash),

2. "I" as the main voice, and "we" for formal arguments and proofs (Spence), and
3. An authorial "we" as the main voice, and imperatives for formal arguments and proofs (Aumann).[27]

I leave it for future exploration to verify this hypothesis on a larger set of texts.

Conclusions

Narrative analysis, I have tried to show in this essay, enriches our understanding of mathematical writing. It provides a window into the nature of mathematical writing, and hence the nature of mathematics more generally.[28] It is through their narrating—the way that mathematicians use language and the narrative code—that mathematicians let us see what they are doing; namely, as I would defend, referring to something. What is referred to, though, is not a concrete, tangible thing but an abstract object. The purpose of mathematical writing, at least practically—no matter what the philosophy of the mathematician at work—consists in making someone "see," with their inner eye, that abstract object and in this way bring it to life.

This is an introductory essay of sorts. Some observations made here only in passing lend themselves to be investigated in separate studies. For instance, as indicated already, it is worth testing on a larger set of mathematical texts whether it is possible to establish a typology based on certain patterns in which multiple voices are combined. But certainly one could also investigate in more detail the role of narrative implicatures in mathematical writing (*What are the silent conventions/maxims of mathematical writing on which such implicatures operate? Can those mechanisms be related to more fine-tuned accounts of quantity implicatures in linguistics?*) or the presence of free indirect style in mathematical writing (*Is this a particularity of applied mathematical text or can it also be demonstrated in pure mathematical text?).*

On the other hand, confronting the concepts of narrative analysis with mathematical text brings out certain new aspects of these concepts, or at least brings them out in a new way—for instance, the difference between functionality as a premise and functionality as a necessary condition, or the notion of *narrative implicature,* or the particular notions of *showing* and *telling* as employed here. Such extensions or variations of the terms of investigation might be useful for the analysis of other kinds of text or genres.

Endnotes

1. This essay is based on a talk that I gave at the workshop *The Limits and Possibilities of Narrative Explanations in Game Theory* at the Wissenschaftskolleg zu Berlin, April 17–18, 2016. I would like to thank Philippe Mongin, one of the organizers of the workshop, Antoine Billot, Michele Odisseas Impagnatiello, Ned Markosian, and Franco Moretti for discussions and comments through different stages of writing this essay. Furthermore, I would like to express my gratitude to Robert

Aumann for his response to my analysis of his "Agreeing to Disagree" and to the editor of this journal for his critique and guidance.

2. This and the following quotes from Barthes's "Introduction to the Structural Analysis of Narrative" (1966) are my own translations from the original French. "Le récit ne fait pas voir, il n'imite pas; la passion qui peut nous enflammer à la lecture d'un roman n'est pas celle d'une 'vision' (en fait, nous ne 'voyons' rien), c'est celle du sens, c'est-à-dire d'un ordre supérieur de la relation, qui possède, lui aussi, ses émotions, ses espoirs, ses menaces, ses triomphes: 'ce qui se passe' dans le récit n'est, du point de vue référentiel (réel), à la lettre: *rien;* 'ce qui arrive,' c'est le langage tout seul, l'aventure du langage, dont la venue ne cesse jamais d'être fêtée" (26–27).

3. The mathematical-objects-as-abstract-artifacts point of view is, to my mind, not necessary to maintain the distinction between mathematical writing as referring to something and relying on truth conditions determined by that something, on the one hand, and fictional narrative as being concerned with the creation of sense, on the other hand. In fact, it seems to me to be a minimal defense. Philosophical accounts of mathematics that do regard mathematical objects as *abstract types,* which as such belong to the real world and only have to be discovered, probably encompass this distinction easily. (Surely, to understand how, it also has to be said what fictional narrative then is.) I do not claim to deliver a philosophical account of what is mathematics and what is fiction. I rather take the above-mentioned distinction as an axiom—use it as an analytical tool—and try to see if anything worth noting with respect to what can be observed in the writing of mathematical text as opposed to fictional narrative can be explained in terms of that distinction.

4. For recent, more general discussions of *arguments that narrate* and *narration as argument,* see Adam, as well as the volumes edited by Danblon et al. and Olmos.

5. Game theory is a branch of applied mathematics that aims at modeling the interaction of individual decision makers. The main quest, in theoretical as well as applied work, is in studying the joint outcome of that interaction. The theory proceeds deductively. A game has to be defined, which includes specifying the rules according to which individuals interact, what their choices are, and how they evaluate the possible outcomes of that interaction. Different solution concepts (equilibrium concepts like Nash equilibrium, or heuristic procedures such as iteratively deleting strategies that are never a best response) have been proposed for different classes of games. Theoretical work is predominantly concerned with studying the formal, mathematical properties of such solution concepts; proposing new solution concepts; studying the relations between different solution concepts; etc. Hence game-theoretic text is a special form of mathematical text. Applied work uses game theory to formally study problems that are the concern of other disciplines (the organization of markets, voting systems, ecological systems, language, etc.). Surely, because game theory is concerned with the interaction of individuals, the interaction of actants in a narrative is also a possible field of application for game theory. Game theorists have explored this idea by using examples from literature to illustrate game-theoretic concepts (see, for example, Brams, "Game Theory" and *Game Theory*). On the other hand, they have also investigated it more explicitly. Steven Brams, for instance, uses game-theoretic methods to analyze stories in the Hebrew Bible (*Biblical Games*). Robert Aumann and Michael Maschler use cooperative game theory to explain a rule for splitting a debt described in the Talmud. Heike Harmgart, Steffen Huck, and Wieland Müller analyze the "miracle" as a randomization device in Richard Wagner's opera *Tannhäuser.* Michael Chwe studies Jane Austen's novels through the lens of game theory.

6. "Où donc chercher la structure du récit? Dans les récits, sans doute" (2); "Comment opposer le roman à la nouvelle, le conte au mythe, le drame à la tragédie (on l'a fait mille fois) sans se référer à un modèle commun?" (1).

7. "Comprendre un récit, ce n'est pas seulement suivre le dévidement de l'histoire, c'est aussi y reconnaître des 'étages,' projeter les enchaînements horizontaux du 'fil' narratif sur un axe implicitement verticale; lire (écouter) un récit, ce n'est pas seulement passer d'un mot à l'autre, c'est aussi passer d'un niveau à l'autre . . . le sens n'est pas 'au bout' du récit, il le traverse" (5–6).

8. It is insightful to look at style recommendations for mathematicians, particularly those written before the widespread use of *LaTeX,* such as the American Mathematical Society's *Manual for Authors of Mathematical Papers* or Paul Halmos's essay "How to Write Mathematics."

9. "dans l'ordre du discours, ce qui est noté est, par définition, notable" (7). This phrase is missing in the 1975 English translation by Barthes and Duisit.

10. "quand bien même un détail paraîtrait irréductiblement insignifiant, rebelle à toute fonction, il n'en aurait pas moins pour finir le sens même de l'absurde ou de l'inutile" (7).

11. See, notably, Bremond, "Le Message" and "La Logique"; Greimas; Lévi-Strauss; and Todorov.

12. "tout segment de l'histoire qui se présente comme le terme d'une corrélation" (7).

13. "La fonction est évidemment, du point de vue linguistique, une unité de contenu: c'est 'ce que veut dire' un énoncé qui le constitue en unité fonctionnelle, non la façon dont cela est dit" (7).

14. "un concept plus ou moins diffus, nécessaire cependant au sens de l'histoire: indices caractériels concernant les personnages, informations relatives à leur identité, notations 'd'atmosphères,' etc." (8–9).

15. Some authors refer to cardinal functions as *turning points* and to catalyzers as *fillers;* see, for example, Moretti 366.

16. "C'est qu'il s'agit ici d'une fonctionnalité purement chronologique (on décrit ce qui sépare deux moments de l'histoire), tandis que dans le lien qui unit deux fonctions cardinales, s'investit une fonctionnalité double, à la fois chronologique et logique: les catalyses ne sont que des unités consécutives, les fonctions cardinales sont à la fois consécutives et conséquentes" (10).

17. "Une notation, en apparence explétive, a toujours une fonction discursive: elle accélère, retarde, relance le discours, elle résume, anticipe, parfois même déroute: le noté apparaissant toujours comme du notable, la catalyse réveille sans cesse la tension sémantique du discours, dit sans cesse : il y a eu, il va y avoir du sens. . . . Disons qu'on ne peut supprimer un noyau sans altérer l'histoire, mais qu'on ne peut non plus supprimer une catalyse sans altérer le discours" (10).

18. It is worth remembering that in French "argument" refers to both the succession of steps in a proof as well as the succession of cardinal functions in a narrative or play.

19. G. E. R. Lloyd, for instance, insists: "To suggest any link between mathematics and narrative would be tantamount to confusing categories. One [mathematics] deals with quantity, the other with time. One sets out timeless truths, the other recounts chronological sequences of actions. A story, such as the plot of a tragedy, as Aristotle famously insisted, deals with a whole, and a whole is defined as having a beginning, a middle, and an end. The facts that it sets out are sequential, even though the order in the narrative may not follow the chronological order of the events narrated" (392).

20. "Les indices impliquent une activité de déchiffrement: il s'agit pour le lecteur d'apprendre à connaître un caractère, une atmosphère; les informants apportent une connaissance toute faite" (11).

21. That this is so even for mathematicians of a different field comes out tellingly from one of the first pieces of advice that the *Manual for Authors of Mathematical Papers* offers: "The first paragraph of the introduction should be comprehensible to any mathematician, and it should pinpoint the location of the subject matter" (2).

22. Commenting on an earlier draft of this essay, Robert Aumann expands: "In other words, the 'implicit assumption' is not really an assumption; it is part of what is meant by state of the world" (Robert Aumann, personal communication, March 24, 2019).

23. Cédric Villani, in *Théorème Vivant,* offers insight into that process of mathematical creation.

24. A well-known example is the "we" that appears on the first page of Flaubert's *Madame Bovary* and then disappears forever.

25. Zadie Smith has recently addressed the question of "plausibility" in fiction.

26. Commenting on an earlier version of this essay, Robert Aumann, explains: "IMHO, in most mathematical writing, 'we' is simply the main narrative voice. The author wishes 'to secure an impersonal style or tone'—speaks as the personification of 'Agreeing to disagree,' not as Mr. Bob. . . . The 'We' that opens the second paragraph of 'Agreeing' ('We publish this observation with some diffidence') should strictly have been 'I.' It is Bob who publishes the observation; the observation does not publish itself" (Robert Aumann, personal communication, March 24, 2019).

27. A further motivation for the choice of these three papers is that their subject matter is potentially of interest for narrative analysis.

28. This is a paraphrase of the title of Steven Pinker's *The Stuff of Thought: Language as a Window into Human Nature* (2007), in which he explores the hypothesis that people's use of language reveals certain regularities in how they structure the world around them—regularities that they themselves are not necessarily aware of.

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